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QUASITRANSVERSE SHOCK WAVES IN ELASTIC MEDIA WITH AN INTERNAL STRUCTURE

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We consider the structure of small-amplitude quasitransverse shock waves in a weakly anisotropic elastic medium which possesses an internal structure generating the wave dispersion. The dispersion is modeled by introducing terms with higher derivatives into the equations of the theory of elasticity, and the dissipation is represented by viscous terms. In one of the two possible cases treated below, the requirement that the discontinuity structure exist leads to a set of admissible discontinuities of complex structure. A considerable part of the shock adiabat consists of a set of short portions and separate points, the number of which increases as the viscosity decreases. This complex set of admissible discontinuities is the general case where the dispersion in the shock-wave structure is sufficiently strong.

Basic Equations. We consider nonlinear quasitransverse waves in an elastic medium with an internal structure (for instance, in a composite). As compared to an ordinary elastic body, whose motion is described by hyperbolic equations, the presence of the internal structure and the corresponding internal geometrical scales results in the appearance of additional terms with higher derivatives in a system of equations which ensure the wave dispersion [1].

To take into account the effect of dissipative processes, the terms which describe the viscosity forces are introduced into the equations of motion. For large-scale phenomena, these additional dispersive and viscous terms are small, and the waves can be described by the equations of the theory of elasticity. However, these terms (see below) can play an important role in describing the structure of discontinuities, in determining the set of admissible discontinuities, and, consequently, in constructing the solutions of the problems.

As in [2, 3], we use the following assumptions:

(a) All the quantities depend on time t and the Cartesian coordinate x_3 (below, the subscript 3 is omitted);

(b) The nonlinearity is weak;

(c) The anisotropy in the wave plane (the wave anisotropy) is weak.

We write the equations of the theory of elasticity for the quasitransverse waves without the dispersive and viscous terms, taking into account that the adopted assumptions eliminate the longitudinal motions from the system of equations. Moreover, this system and the discontinuity relations are written in the form of the equations of motion for an equivalent incompressible medium:

$$\rho_0 \frac{\partial v_i}{\partial t} - \frac{\partial}{\partial x}\Big(\frac{\partial H}{\partial u_i}\Big) = 0, \qquad \frac{\partial u_i}{\partial t} - \frac{\partial v_i}{\partial x} = 0, \quad i = 1, 2; \tag{1}$$

$$\rho_0 W[v_i] + \Big[\frac{\partial H}{\partial u_i}\Big] = 0, \qquad [v_i] + W[u_i] = 0,$$
$$H = \frac{1}{2} f(u_1^2 + u_2^2) + \frac{1}{2} g(u_2^2 - u_1^2) - \frac{1}{4}æ(u_1^2 + u_2^2)^2. \tag{2}$$

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Here $u_i = \partial w_i/\partial x$ (w_i is the displacement of the medium along the Cartesian x_i axis orthogonal to the x axis) and W is the discontinuity velocity; the constants $g > 0$ and æ characterize the anisotropy and nonlinearity, respectively; the terms in Eqs. (1) and (2) containing these constants are assumed to be small in comparison with other terms, which correspond to a linear isotropic medium, and the square brackets in (2) denote the jump of the bracketed quantity: the difference between the values behind and before the jump.

Equations (1) with discontinuity conditions (2) is a hyperbolic model that describes large-scale phenomena.

In passing to a complete system of equations that takes into account the dispersion and dissipation, we use the additional assumption:

(d) The dispersive and viscous terms are small and do not exceed the nonlinear terms in order of magnitude for the solutions considered.

As factors the dispersive terms contain the higher derivatives of u_i with respect to x and since they are assumed to be small, we neglect the change in their coefficients, i.e., we will consider these terms in the linear approximation. Moreover, we confine ourselves to terms containing derivatives with respect to x of order not greater than the third order, since in the longwave (compared to the internal geometrical scale of the medium) solutions considered below, the magnitudes of the derivatives decrease as their order increases. The longwave character (the large spatial scale) of the solutions under consideration is due to the small nonlinearity. This property will be obvious in treating the problem of the structure of discontinuities.

It follows from [1] that the dispersive terms can be represented by introducing, into the first pair of equations (1), terms with third-order derivatives of u_i with respect to x or terms with second-order derivatives, which can enter the equations of motion in the form $b_{ij}u_j''$, where $u_j'' = \partial^2 u_j/\partial x^2$, and b_{ij} is a 2×2 asymmetric matrix (precisely this form of matrix ensures dispersion in the absence of dissipation). The terms $b_{ij}u_j''$ are the components of the two-dimensional vector. For the asymmetric matrix b_{ij}, this vector can be written as $\boldsymbol{u}'' \times \boldsymbol{b}$, where $\boldsymbol{u} = u_1\boldsymbol{e}_1 + u_2\boldsymbol{e}_2$ and $\boldsymbol{b}$ is the pseudovector directed along the x axis. We note that the expression $\boldsymbol{u}'' \times \boldsymbol{b}$ is the vector in the only case where $\boldsymbol{b}$ is the pseudovector.

Thus, for the dispersion terms with second-order derivatives to enter the equations of motion, it is necessary to introduce the pseudovector in the formulation of the problem. This situation arises during the propagation of a nonlinear electromagnetic wave in a magnet [4] when the initial magnetic field can be considered as a pseudovector.

In this paper, we consider the case where there are no pseudovectors in the formulation of the problem, so that the dispersive terms are represented by terms with third-order derivatives $m_{ij}\partial^3 u_j/\partial x^3$, where $m_{ij} =$ const. If the wave anisotropy is ignored, they can be written in the form $m\partial^3 u_i/\partial x^3 = m\partial^4 w_i/\partial x^4$.

It should be noted that terms of this type with $m > 0$ enter into the left-hand sides of the equations of motion, for example, in the case where a readily deformable homogeneous elastic medium contains uniformly distributed rigid rods with sufficient flexural rigidity which are parallel or make a small angle with the x axis.

Moreover, we assume that the dissipative processes are represented by the viscous terms $\mu\partial^2 v_i/\partial x^2$ ($\mu = \text{const} > 0$), which enter the left-hand side of the first group of Eqs. (1) with the minus sign. The viscous terms are assumed to be small and, therefore, we write them in isotropic form. Thus, the complete system of equations suitable for describing the discontinuity structure can be written in the form

$$\rho_0 \frac{\partial v_i}{\partial t} - \frac{\partial}{\partial x}\Big(\frac{\partial H}{\partial u_i}\Big) + m\frac{\partial^3 u_i}{\partial x^3} - \mu\frac{\partial^2 v_i}{\partial x^2} = 0, \qquad \frac{\partial u_i}{\partial t} - \frac{\partial v_i}{\partial x} = 0. \tag{3}$$

***A Priori* Evolution. The Problem of the Shock-Wave Structure.** For the case $m = 0$, i.e., dispersion is absent and only the viscosity is taken into account, the structure of shock waves was studied in [3, 5]. It was shown that the structure corresponds to all *a priori* evolutionary discontinuities. We call the discontinuities which are evolutionary [6] under the assumption that they satisfy only (2), *a priori* evolutionary relations. The evolutionary conditions are written in the form of a system of inequalities between the shock-wave velocity W and the velocities of the characteristics of system (1) c_1^- and c_2^- and c_1^+ and c_2^+ in front of and behind the discontinuity.

The evolutionary conditions are shown in Fig. 1, where the velocity values are laid off along the axes. Each point in the diagram corresponds to the discontinuity, for which certain inequalities for W, $c_{1,2}^-$, and $c_{1,2}^+$

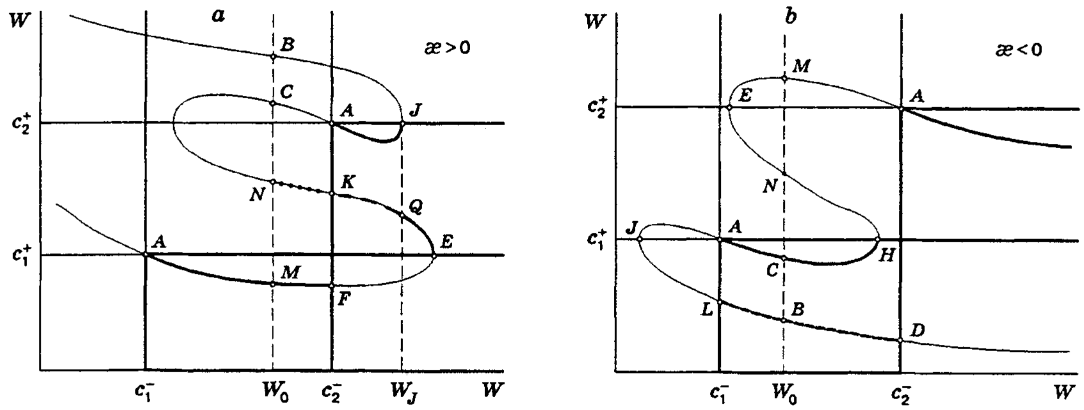

Fig. 1

are satisfied. The points in the bold rectangles refer to the *a priori* evolutionary discontinuities; moreover, the left bottom rectangle corresponds to slow shock waves, and the right upper rectangle to fast shock waves. The shock adiabat calculated by relations (2) is mapped onto the evolutionary diagram [2, 3].

We consider the effect of dispersion on the structure and the set of admissible discontinuities. We find the bounded solutions of system (3), which depend on $\xi = x - Wt$. After a single integration of each equation from the first pair of equations and the elimination of v_i, system (3) takes the form

$$m u_i'' + \mu W u_i' = -\frac{\partial F}{\partial u_i} \qquad (i = 1, 2),$$
$$F = \frac{1}{2}(\rho_0 W^2 - f)(u_1^2 + u_2^2) - \frac{1}{2}g(u_2^2 - u_1^2) + \frac{æ}{4}(u_1^2 + u_2^2)^2 + A_1 u_1 + A_2 u_2, \tag{4}$$

where A_i are the integration constants, which can be found from the condition that $\partial F/\partial u_i = 0$ for the state in front of the discontinuity $u_i = u_i^-$, and the primes denote derivatives with respect to ξ.

Equations (4) have the form of the equations of motion of a material point of mass m with the friction proportional to the velocity in the potential force field determined by the potential energy F. The variable ξ plays the role of time. We use this analogy to obtain certain qualitative inferences on the structure and the set of admissible discontinuities. A more detailed investigation requires numerical methods.

It is worth noting that W can be found from the relation for discontinuity (2) independently of the existence of the structure and, consequently, the parameters m and μ, which determine this structure.

For a specified function F and value of W, the integral curves of system (4) and, consequently, the existence of the structure are determined by the ratio $\mu/\sqrt{m}$, which characterizes the relation between the viscous and dispersive terms in the equations. For simplicity, we confine ourselves to the case of small values of $G = g/R^2$, where $R^2 = (u_1^-)^2 + (u_2^-)^2$.

The case $æ > 0$. If we set $g = 0$ and $A_i = 0$, the function $F(u_1, u_2)$ for $\rho_0 W^2 < f$ has the form of a ring groove, whose outer walls go upward unlimitedly, and, for $u_1 = 0$ and $u_2 = 0$, the local maximum occurs. If $g \neq 0$ and $A_i = 0$, the groove depth is variable and the groove has two local minima separated by two saddle points. The local maximum in the central part is preserved. For small A_i, the number and type of stationary point do not change.

The stationary points of the function $F(u_1, u_2)$, together with equalities $u_i' = 0$, determine the singular points of system (4) and can correspond to the states for $\xi = \pm\infty$. The cases where there are five stationary points are of special interest. This situation can occur when, for example, $c_1^- < W < c_2^-$. In this case, the vertical line $W =$ const in Fig. 1a can intersect the shock adiabat at four points: B, C, N, and M. In the u_1, u_2 plane, the points B, C, N, and M of the shock adiabat correspond to the stationary points of the function F (Fig. 2). Moreover, there is a stationary point A which refers to the initial state (in Fig. 2, the groove is shown by the dashed curve). In the case under consideration, the point M is the maximum point, B and C are the minimum points, and N and A are the saddle points. The projection of the integral curves belonging to the

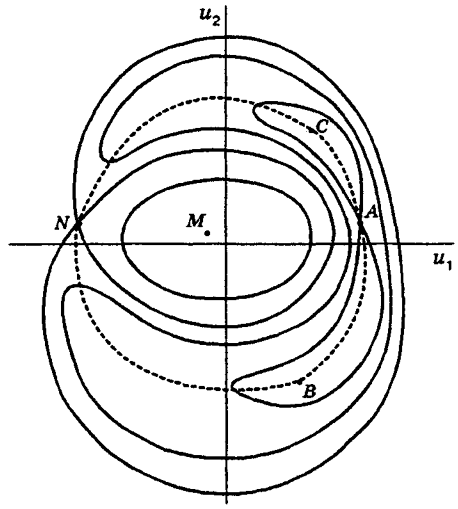

Fig. 2

four-dimensional space u_1, u_2, u_1', and u_2' on the u_1, u_2 plane are the trajectories of motion of a material point which moves with friction over the surface given by the graph $F(u_1, u_2)$. The discontinuity structure with a state before the discontinuity which is specified by the point A is represented by the trajectory that arrives at the point A as $\xi \to \infty$ and originates with zero velocity $u_1' = 0$ and $u_2' = 0$ from one of the critical points B, C, M, and N, the last point corresponding to the state behind the discontinuity.

Obviously, the integral curves which originate from the minimum points of the potential energy B or C with zero velocities $u_1' = 0$ and $u_2' = 0$ cannot exist, i.e., the structure of the nonevolutionary shock waves A $\to$ B and A $\to$ C does not exist.

We consider a set of trajectories originating from the maximum point M. This set depends on one parameter. All the trajectories should arrive at one of the points A, B, C, and N as $\xi \to \infty$. The set of trajectories entering the minimum point B and the set of trajectories entering the minimum point C depend on one parameter as well. This implies the existence of integral curves which separate these sets and enter the points A and N. Thus, the structure of the slow shock wave A $\to$ M exists.

We ascertain whether an integral curve from N to A with increasing ξ exists. To do this, the inequality $F(\mathrm{N}) > F(\mathrm{A})$ must be satisfied, which is the case if W is greater than a certain W_0. In this case, the points A and C are close to each other, while the point N lies in the distance and, consequently, higher than A (Fig. 2). If W is close to c_1^-, the points A and M become closer and, as a result, the point A is higher than the point N and the integral curve from N to A. Consequently, the structure of the shock wave A $\to$ N does not exist.

We now consider the situation shown in Fig. 2, by analogy with the motion of a heavy material point. The heavy points which leave the point N with zero velocity move along the groove and, generally, they stop at the minimum B or C. If the effect of friction characterized by the ratio $\mu/\sqrt{m}$ is sufficiently weak, the heavy point performs oscillations along the groove, every time passing by the point A. If the parameters of the equations change, for instance, the velocity W, the number of oscillations changes and one can indicate the alternating intervals on the W axis which correspond to the termination of the motion at the points B and C. These intervals are separated by the values of W such that the heavy point under consideration attains neither B nor C as $\xi \to \infty$. In this case, only one possibility exists: the heavy point moves toward the unstable (saddle) stationary point A as $\xi \to \infty$. This means that, for the indicated values of W, the structure of the shock wave A $\to$ N exists. As shown in Fig. 1a, the shock wave A $\to$ N is *a priori* nonevolutionary. It becomes evolutionary by virtue of the fact that the specified value of the velocity at which it exists should be regarded as an additional condition at the discontinuity (similar to the combustion front in a gas).

For small values of the friction coefficient, the number of oscillations of the heavy point is great and the slight change in W is sufficient for the point of stop to change from B to C or vice versa. Moreover, W passes through a certain value which corresponds to the stoppage at the point A. In other words, for small $\mu/\sqrt{m}$, the distance between these values of W at the W axis, at which the structure of the shock wave A $\to$ N exists, is small. These values belong to the section which adjoins the point $W = c_2^-$ from the left (see Fig. 1a).

For $W > c_2^-$, the qualitative pattern of the level curves remains similar to that shown in Fig. 2; however, the points A and C interchange. Moreover, the structure of the fast shock wave A $\to$ C (the upper fast branch of the shock adiabat in Fig. 1a) always exists. The structure of the discontinuity A $\to$ B does not exist, while the structure of the discontinuity A $\to$ M is not unique, which is typical of the dissociating nonevolutionary discontinuities.

The discontinuities A $\to$ N for $W > c_2^-$ possess the structure if the velocity W belongs to one of the intervals of the W axis, the length of which tends to zero and the number tends to infinity as $\mu \to 0$. This conclusion is similar to the above conclusion that the integral curve which connects the points C and N exists for $c_1^- < W < c_2^-$. However, one should keep in mind that two integral curves emanate from the point N and, for this case, in order that the structure of the shock wave A $\to$ N be absent, it is necessary that both curves arrive at the point B. One can expect that this situation occurs for sufficiently small μ in the case where W is close to c_2^-. Moreover, the points C and A are close to each other, and the groove depression in the neighborhood of the point C is small.

For $W > W_J$, the groove has only one depression with the deepest point A. In this case, both integral curves come from point N to point A and the structure of the shock wave A $\to$ N exist.

For the case $æ > 0$, the set of all admissible discontinuities at the shock adiabat is shown in Fig. 1a by bold sections and points; the number of separate points and short sections depends on the parameters μ and m determining the structure via the ratio $\mu/\sqrt{m}$.

The case $æ < 0$. For small values of g and A_i, the graph of the function $F(u_1, u_2)$ resembles the surface of the volcano's crater. If the value of W corresponds to the right vertical dashed line in Fig. 1b, we have five stationary points of the function $F(u_1, u_2)$. The pattern of the level curves is similar to that shown in Fig. 2; however, the direction of change of the function F from curve to curve is opposite. Moreover, the point M is the minimum point, B and C are the maximum points, and A and N are the saddle points.

In the case of small values of g, where the height of the crater's walls changes slightly, among the trajectories emanating from the point of the lesser maximum (the point C in Figs. 2 and 3), there exist two bundles of trajectories which pass along the crater's ridge (the dashed curve in Figs. 2 and 3) in two opposite directions. When the heavy particle deviates at a certain distance from the ridge, it undergoes significant transversal acceleration and departs. Each of the bundles encounters an obstacle in the form of the neighborhood of the second minimum. Here, the bundles are separated, so that one part goes inside, while another goes outside the crater. It is obvious that, in this case, the trajectories which connect the points C and A and which C and N exist. The first trajectory represents the structure of the slow shock wave.

We consider the trajectories which emanate from the point of the greater maximum B. As in the previous case, for small G, two bundles of trajectories which go along the ridge exist. In each bundle, there is a trajectory which must end up at the ridge at the point A or N after a certain number of oscillations along the ridge. If at least one of the trajectories ends up at the point A, this implies that the structure of the slow shock wave A $\to$ B exists. However, one can predict that when the velocity W changes, the intervals of its values can be found where both trajectories end up at the point N. In this case, there is an interval on the shock adiabat which does not correspond to the admissible shock waves. The length of these intervals and the distance between them tend to zero as $\mu/\sqrt{m} \to 0$.

We now pass to the existence of the structure of the "intermediate" shock wave A $\to$ N. Obviously, this structure is absent for $F(\mathrm{N}) < F(\mathrm{A})$, i.e., for $W < W_*$. For increasing of W depending on U_1 and U_2, a value W_0 may be found such that, for $W > W_0$, a single trajectory going toward the crater intersects the ridge at the lower part in the neighborhood of the point A and goes into the external domain. For this value of $W = W_0$, a trajectory N $\to$ A, which corresponds to the structure of the intermediate shock wave A $\to$ N, exists.

The aforesaid implies the set of admissible discontinuities, which are shown in the evolution diagram by bold sections and the point (see Fig. 1b).

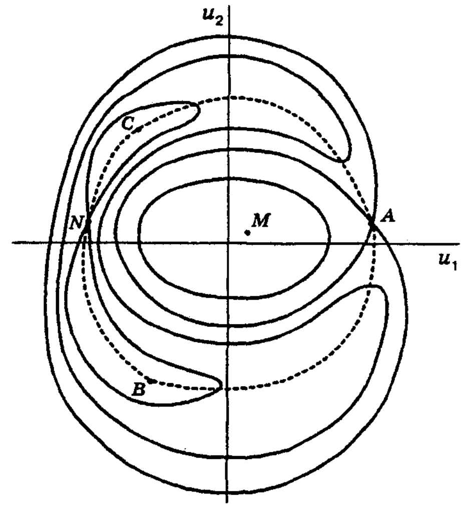

Fig. 3

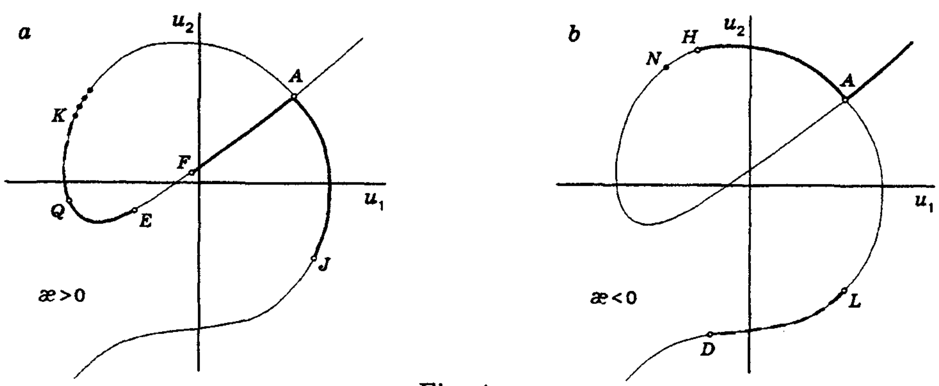

Fig. 4

In conclusion, we show the shock adiabat with a displayed set of admissible discontinuities in the u_1, u_2 plane (Fig. 4). As in the case of electromagnetic shock waves, the set of admissible intermediate shock waves, which occurs for $æ > 0$, leads to the nonuniqueness of the solutions of the problems.

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